THE EXISTENCE OF BOUNDED SOLUTIONS TO THE COMPLEX MONGE-AMPÈRE TYPE EQUATIONS ON HERMITIAN MANIFOLDS WITH BOUNDARY

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Abstract: *In the paper, we will study and discuss about the existence of bounded solution to the complex Monge-Ampère type equation* $F(u,z)d\mu = (\omega + dd^c u)^n$ *on Hermitian manifold* $(\overline{X},\omega)$ *with nonempty boundary, where* u *is an unknown function,* μ *is a nonnegative Radon measure on* X *and* $F(t,z)$ *is a* $dt \times d\mu$ *-measurable function.*
Keywords: *bounded solutions, bounded subsolutions, complex Monge-Ampère type equation, Hermitian manifolds, ω-plurisubharmonic functions.*

Nhận bài ngày 10.04.2025; gửi phản biện, chỉnh sửa, duyệt đăng ngày 30.05.2025
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1. INTRODUCTION

Let $(\overline{X},\omega)$ be a Hermitian manifold with $dim_{\mathbb{C}} X = n \geq 1$, nonempty boundary ∂X and a fixed background Hermitian metric ω on $\overline{X}$. In this paper, we always assume that the Hermitian metric ω satisfies the following conditions:

$$-B\omega^2 \leq 2n dd^c\omega \leq B\omega^2; -B\omega^3 \leq 4n^2 d\omega \wedge d^c\omega \leq B\omega^3 \tag{1}$$

in $\overline{X}$, where, $B > 0$ only depends on ω, n; we use differential operators $d = \partial + \overline{\partial}$, $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$; then $dd^c = \frac{i}{\pi}\partial\overline{\partial}$ $dd^c = \frac{i}{\pi}\partial\,\bar{\partial}$.

By $PSH(\omega,X)$ we denote set of ω -plurisubharmonic functions on X. Let μ be a nonnegative Radon measure on X. Assume that $F: \mathbb{R} \times X \to [0,+\infty)$ is a $dt \times d\mu$ -measurable function. We consider the following Monge-Ampère type equation

$$\begin{cases} u \in PSH(\omega,X) \cap L^\infty(X); \\ (\omega + dd^c u)^n = F(u,\cdot)d\mu; \end{cases} \tag{2}$$

where u is an unknown function and $(\omega + dd^c u)^n$ stands for the complex Monge-Ampère operator of u.

When X is a bounded domain in $\mathbb{C}^n$ and ω is the zero form, the equation (2) is the Monge-Ampère type equations for plurisubharmonic functions in X. In this case, solutions of (2) are not only considered in the class of bounded functions but also extended to unbounded functions. There are many results on solving this equation due to [1], [2], [3], [12], etc.

On a Hermitian manifold $(\overline{X},\omega)$ with nonempty boundary, if $F(t,z) \in C^\infty(\mathbb{R} \times X)$ and $\mu = \omega^n$ such that $F(v,z) \leq (\omega + dd^c v)^n$ with some smooth ω -plurisubharmonic function v in X, the authors in [6] proved that the equation (2) exists a smooth solution. When $F(t,z) = f(z) \in L^p(X), p > 1$ and μ is a smooth measure, the authors in [10] showed that there exists a continuous solution of the equation (2). Recently, the authors in [7] and [8] obtained some results on the existence of bounded solution of the equation (2). Namely, when $F(t,z) = 1$and $\mu \leq (\omega + dd^c v)^n$, with some $v \in PSH(\omega,X) \cap L^\infty(X)$ such that the restriction of v in ∂X is a continuous function, the authors in [7] proved that the euquation (2) has a bounded solution u such that $u = v$ in ∂X. In the case that $F(t,z)$ is bounded, the authors in [8] also showed the existence of bounded solution of (2) when $F(t,z)$ satisfies that there exists a function $v \in PSH(\omega, X) \cap L^\infty(X)$ such that $F(v,\cdot)d\mu \leq (\omega + dd^c v)^n$. Also in [8], the authors gave the proof of the existence of bounded solution to the equation (2) when $F(t,z) = e^{\lambda t}$ and $\mu \leq (\omega + dd^c v)^n$ with some $v \in PSH(\omega, X) \cap L^\infty(X)$. Continuing the above direction of research, in this paper, we wish to investigate the existence of bounded solutions to (2) in case of the arbitrary function $F(t,z)$.

Now we give the main result of our paper which is inspired from the results in [3] and [8].

Theorem Let $(\overline{X} = X \cup \partial X, \omega)$ be an n -dimensional Hermitian manifold such that the boundary ∂X is nonempty and there exists a function $f \in PSH(\omega, X) \cap C(X)$ satisfying that $(\omega + dd^c f)^n = 0$. Let μ be a finite non-negative Radon measure in X satisfying that at each point $z \in X$, there exist a neighborhood U_z of z and a function $v_z \in PSH(U_z) \cap L^\infty(U_z)$ such that $\mu \leq (dd^c v_z)^n$. Assume that $F: \mathbb{R} \times X \to [0, +\infty)$ is a $dt \times d\mu$ -measurable function such that:

(1) For all $z \in X$, the function $t \mapsto F(t,z)$ is continuous and non-decreasing.

(2) At each $z \in X$, there exist a neighborhood W_z of z and a function $w_z \in PSH(W_z) \cap L^\infty(W_z)$ such that $F(t,\cdot)d\mu \leq (dd^c w_z)^n, \forall t \in \mathbb{R}$, in W_z.

(3) There exists $u \in PSH(\omega, X) \cap L^\infty(X)$ such that $F(u,\cdot)d\mu \leq (\omega + dd^c u)^n$ and $u \leq f$.

Then there exists a function $w \in PSH(\omega, X) \cap L^\infty(X)$ such that $F(w,\cdot)d\mu = (\omega + dd^c w)^n$ and $w \leq f$.

The paper is organized as follows. In Section 2, we recall some notions of ω -plurisubharmonic functions which are necessary for the next results of the paper; after we give the proof of Theorem 1.1. Section 3 gives conclusion of paper.

2. CONTENT

2.1. Preliminaries

In this section, we recall some elements of theory of ω -plurisubharmonic functions that will be used throughout the paper. These results can be found in [4], [5], [9], [10]. In this paper, by $PSH(\Omega)$ we denote set of plurisubharmonic functions in open set $\Omega \subset \mathbb{C}^n$. By $C(X)$ we denote class of continuous functions in X.

Now, we recall the definition ofω -plurisubharmonic functions in open subsets of $\mathbb{C}^n$. Let Ω be an open subset in $\mathbb{C}^n$ and ω be a Hermitian metric on $\mathbb{C}^n$. We have the following definition.

Definition 2.1. Let a function $u: \Omega \to [-\infty, +\infty)$ be an upper semi-continuous function. Then u is called ω -plurisubharmonic (ω -psh for short) if $u \in L^1_{loc}(\omega^n, \Omega)$ and $\omega + dd^c u \geq 0$ as a current. By $PSH(\omega, \Omega)$ we denote set of ω -psh functions in Ω.

Note that for each $u \in PSH(\omega, \Omega)$ and each point $z \in \Omega$, there exist a neighborhood $U \subset \Omega$ of z and a smooth plurisubharmonic function ρ in U such that $\omega \leq dd^c \rho$ in U and $u + \rho \in PSH(U)$.

Next, let (X, ω) be an n -dimensional Hermitian manifold. We have the following definition.

Definition 2.2. An upper semi-continuous function $u: X \to [-\infty, +\infty)$ is called ω -plurisubharmonic if $u \in L^1_{loc}(\omega^n, X)$ and $\omega + dd^c u \geq 0$ as a current. By $PSH(\omega, X)$ we also denote set of ω -psh functions in X.

Note that $u \in PSH(\omega, X)$ if and only if $u \in PSH(\omega, \Omega)$ for arbitrary coordinate chart $\Omega \subset X$.

Finally, we give the introduction of the complex Monge - Ampère operator of ω - plurisubharmonic functions. Fix an open subset $\Omega \subset \mathbb{C}^n$ and a Hermitian metric ω on $\mathbb{C}^n$. Let $u_1, \dots, u_k \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$, for $k \in [1, n]$. By the arguments in [4] and [10], the wedge product $(\omega + dd^c u_1) \wedge \dots \wedge (\omega + dd^c u_k)$ is defined as a nonnegative (k, k) -current in Ω.

When $k = n$ and $u_1 = \dots = u_n = u$, the wedge product

$$(\omega + dd^c u)^n := \underbrace{(\omega + dd^c u) \wedge \dots \wedge (\omega + dd^c u)}_{n \quad times}$$

is called complex Monge-Ampère operator of u. This operator is also defined as a nonnegative Radon measure in Ω Ω. Furthermore, if $u_j \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$ is either uniformly convergent or monotonely convergent almost everywhere to $u \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$, then $(\omega + dd^c u_j)^n \to (\omega + dd^c u)^n$ in the sense of currents.

Now, let $(\overline{X}, \omega)$ be an n -dimensional Hermitian manifold. Take $u \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$. Then $u \in PSH(\omega, \Omega)$ for any coordinate chart $\Omega \subset\subset X$. By using partition of unity, we define the complex Monge-Ampère operators $(\omega + dd^c u)^n$ of u. Moreover, $(\omega + dd^c u)^n$ is a nonnegative Radon measure in X. It is also clear that $(\omega + dd^c u_j)^n \to (\omega + dd^c u)^n$ in the sense of currents for every sequence $\{u_j\} \subset PSH(\omega, X) \cap L^\infty(X)$ satisfying that u_j converges either uniformly or monotonely a.e. to $u \in PSH(\omega, X) \cap L^\infty(X)$.

2.2. The existence of bounded solutions to the complex Monge-Ampère type equations

In this section, we will establish a result on the sovablity of the equation (2). Namely, the content of this result is expressed by Theorem 1.1 in Section 1 and we will give its proof here.

Firstly, we give a local result which is inspired form the results in [3] and [8] and used for the proof of Theorem 1.1. Fix a strictly pseudoconvex domain Ω in $\mathbb{C}^n$. Fix a Hermitian metric ω in neighborhood U of $\overline{\Omega}$ such that in U, the condition (1) is still valid for ω. Let φ be a continuous function on $\partial\Omega$. Theorem 4.2 and Corollary 3.4 in [10] follows that the following equation

$$\begin{cases} h \in PSH(\omega, \Omega) \cap L^\infty(\Omega); \\ (\omega + dd^c h)^n = 0, \qquad in \quad \Omega; \\ h = \varphi, \qquad on \quad \partial\Omega; \end{cases}$$

always exists a unique solution h. For convenience, by h_φ we denote solution of the above equation. We have the following lemma.

Lemma 2.3. Let a finite nonnegative Radon measure μ be such that $\mu \le (dd^c v)^n$ in Ω, for some $v \in PSH(\Omega) \cap L^\infty(\Omega)$. Let φ be a continuous function on $\partial\Omega$. Let $F: \mathbb{R} \times \Omega \to [0, +\infty)$ be a $dt \times d\mu$ -measurable function such that:

(1) For all $z \in \Omega$, the function $t \mapsto F(t, z)$ is continuous, nondecreasing.
(2) For all $t \in \mathbb{R}$, the function $z \mapsto F(t, z)$ is in $L^1(d\mu, \Omega)$.
(3) There exists $w \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$ such that $F(h_\varphi, \cdot) d\mu \le (\omega + dd^c w)^n$ and $w = \varphi$ on $\partial\Omega$.