NONEXISTENCE RESULT TO DOUBLE PHASE PROBLEMS INVOLVING Δ_λ-LAPLACIAN WITH EXPONENTIAL NONLINEARITY

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Abstract: *In this paper, we consider the instability of weak solutions to the problem*

$$-\operatorname{div}_\lambda(w(x)|\nabla_\lambda u|^{p-2}\nabla_\lambda u + \widetilde{w}(x)|\nabla_\lambda u|^{q-2}\nabla_\lambda u) = f(x)e^u \text{ in } \mathbb{R}^N,$$

where $\nabla_\lambda = (\lambda_1\partial_{x_1}, \lambda_2\partial_{x_2}, \ldots, \lambda_N\partial_{x_N}), \lambda_i, i = 1, \ldots, N$ *satisfy some general hypotheses,* $q \geq p \geq 2$ *and* $w, \widetilde{w}, f \in L^1_{\text{loc}}(\mathbb{R}^N)$ *are nonnegative functions satisfying some growth conditions at infinity. Our results can be seen as a generalization of that from the Laplace operator to the* Δ_λ*-Laplacian. Moreover, our result is also extension of that from the Grushin operator to the* Δ_λ*-Laplacian.*

Key words: *Double phase problem, exponential nonlinearity, nonexistence result, stable solutions,* Δ_λ*-Laplacian.*

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1. INTRODUCTION AND MAIN RESULT

In this paper, we define $\nabla_\lambda = (\lambda_1\partial_{x_1}, \lambda_2\partial_{x_2}, \ldots, \lambda_N\partial_{x_N}), \operatorname{div}_\lambda = \nabla_\lambda \cdot, \lambda_i, i = 1,2, \ldots, N$ satisfy some general hypotheses introduced in [1,2]:

(H1) There is a group of dilations $(\delta_t)_{t>0}$

$$\delta_t: \mathbb{R}^N \to \mathbb{R}^N, (x_1, \ldots, x_N) \mapsto (t^{\varepsilon_1}x_1, \ldots, t^{\varepsilon_N}x_N),$$

where $1 = \varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_N$, such that

$$\lambda_i(\delta_t(x)) = t^{\varepsilon_i - 1}\lambda_i(x)$$

for all $x \in \mathbb{R}^N, t > 0$ and $i = 1,2, \ldots, N$.

(H2) $\lambda_1 = 1, \lambda_i(x) = \lambda_i(|x_1|, \ldots, |x_{i-1}|)$ for $i = 2, \ldots, N$ and these functions are continuous on $\mathbb{R}^N$, strictly positive and of class C^1 on $\mathbb{R}^N \setminus \Pi$, where

$$\Pi = \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N : \prod_{i=1}^N x_i = 0\right\}.$$

Then, the homogeneous dimension of $\mathbb{R}^N$ associated to the group of dilations $(\delta_t)_{t>0}$ is

$$Q = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_N \tag{1}$$

and the Δ_λ-norm of $x \in \mathbb{R}^N$ is defined by

$$|x|_\lambda = \left(\sum_{j=1}^{N} \prod_{i \neq j} \lambda_i^2 \varepsilon_j^2 x_j^2 \right)^{\frac{1}{2\sigma}},$$

where $\sigma = 1 + \sum_{i=1}^{N} (\varepsilon_i - 1)$.

The purpose of this paper is to establish nonexistence result of stable weak solutions for the problem

$$-\operatorname{div}_\lambda(w(x)|\nabla_\lambda u|^{p-2}\nabla_\lambda u + \widetilde{w}(x)|\nabla_\lambda u|^{q-2}\nabla_\lambda u) = f(x)e^u \text{ in } \mathbb{R}^N \quad (2)$$

Here, $q \geq p \geq 2$ and $w, \widetilde{w}, f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative functions satisfying

$$\begin{cases} w(x) \leq C_1 |x|_\lambda^a, \\ \widetilde{w}(x) \leq C_2 |x|_\lambda^{\tilde{a}}, \\ f(x) \geq C_3 |x|_\lambda^b \end{cases} \quad (3)$$

for all $|x|_\lambda > R_0$, where $R_0, C_1, C_2, C_3 > 0$ and $a, \tilde{a}, b \in \mathbb{R}$.

Considering the special case $\lambda_i = 1$. If $\widetilde{w} = 0, p = 2$, then (2) becomes the Laplace-type equation. In [3,4], the authors established nonexistence results of stable solutions for the problem

$$-\Delta u = e^u \text{ in } \mathbb{R}^N.$$

In [5], Wang and Ye proved that the equation

$$-\Delta u = |x|^b e^u \text{ in } \mathbb{R}^N,$$

where $b > -2$, does not have stable solutions if $N < 10 + 4b$. If $\widetilde{w} = 0, p \geq 2$, then (2) becomes the *p*-Laplace-type equation. The results about the instability of solutions for the equation

$$-\Delta_p u = f(x)e^u \text{ in } \mathbb{R}^N$$

can be found in [6]. For the more general *p*-Laplace-type equations, we refer to [7–9]. If $\widetilde{w} > 0, q > p \geq 2$, then (2) becomes the double phase problem. In particular, under the condition

$$N < \frac{\min\{p, q - \tilde{a}\}(q+3) + 4b}{q-1},$$

Phuong Le [10] proved that the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u + \widetilde{w}(x)|\nabla u|^{q-2}\nabla u) = f(x)e^u \text{ in } \mathbb{R}^N$$

has no stable weak solutions.

Now, we consider the general case of λ_i. In the last decades, there have been many studies on PDEs involving the strongly degenerate elliptic operators $\Delta_\lambda = \operatorname{div}_\lambda \circ \nabla_\lambda$, see e.g. [11–18]. They were first introduced by Franchi and Lanconelli in [1] and named Δ_λ-Laplacians by Kogoj and Lanconelli in [2]. However, to the best of our knowledge, there has no work dealing with Equation (2) in the general case of λ_i. Hence, the purpose of this paper is to establish some results about the instability of weak solutions for Equation (2), where λ_i,

$i = 1, 2, \ldots, N$ satisfy the hypotheses (H1, H2), $q \geq p \geq 2$ and $w, \widetilde{w}, f \in L^1_{\text{loc}}(\mathbb{R}^N)$ are nonnegative functions satisfying (3).

To state our result, we introduce a functional class of solutions for Equation (2). Let Ω be an open domain in $\mathbb{R}^N$ and put

$$\begin{array}{rcl} H\colon \Omega \times [0,\infty) & \longrightarrow & [0,\infty) \\ (x,t) & \longmapsto & w(x)t^p + \widetilde{w}(x)t^q. \end{array}$$

Define

$$\rho_H(u) = \int_\Omega H(x, |u|) = \int_\Omega (w(x)|u|^p + \widetilde{w}(x)|u|^q)$$

and

$$L^H(\Omega) = \{u\colon \Omega \to \mathbb{R} \mid u \text{ is measurable and } \rho_H(u) < \infty\}.$$

We define the norm in the space $L^H(\Omega)$ as follows

$$\|u\|_H = \inf\left\{\tau > 0 \mid \rho_H\left(\frac{u}{\tau}\right) \leq 1\right\}.$$

The corresponding Sobolev space

$$W^{1,H}(\Omega) = \{ u \in L^H(\Omega) \mid |\nabla_\lambda u| \in L^H(\Omega)\}$$

is equipped the norm

$$\|u\|_{1,H} = \||\nabla_\lambda u|\|_H + \|u\|_H.$$

We set

$$W^{1,H}_{\text{loc}}(\Omega) = \left\{u\colon \Omega \to \mathbb{R} \mid u\varphi \in W^{1,H}_0(\Omega) \text{ for all } \varphi \in C^1_c(\Omega)\right\},$$

where $W^{1,H}_0(\Omega)$ is the closure of $C^1_c(\Omega)$ in $W^{1,H}(\Omega)$ with respect to the $\|.\|_{1,H}$ norm.

In this paper, the solutions of Equation (2) are understood in the sense of weak solutions as follows.

Definition 1.1. *A function $u \in W^{1,H}_{\text{loc}}(\mathbb{R}^N)$ is called a weak solution of Equation* (2) *if $f(x)e^u \in L^1_{\text{loc}}(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} (w(x)|\nabla_\lambda u|^{p-2}\nabla_\lambda u + \widetilde{w}(x)|\nabla_\lambda u|^{q-2}\nabla_\lambda u) \cdot \nabla_\lambda \varphi = \int_{\mathbb{R}^N} f(x)e^u \varphi \tag{4}$$

for all $\varphi \in C^1_c(\mathbb{R}^N)$.

We now define the stable weak solutions of Equation (2).

Definition 1.2. *A weak solution u of Equation* (2) *is called stable if*

$$\begin{aligned} &\int_{\mathbb{R}^N} w(x)[|\nabla_\lambda u|^{p-2}|\nabla_\lambda \varphi|^2 + (p-2)|\nabla_\lambda u|^{p-4}(\nabla_\lambda u \cdot \nabla_\lambda \varphi)^2] \\ &\qquad + \int_{\mathbb{R}^N} \widetilde{w}(x)[|\nabla_\lambda u|^{q-2}|\nabla_\lambda \varphi|^2 \\ &\qquad\qquad + (q-2)|\nabla_\lambda u|^{q-4}(\nabla_\lambda u \cdot \nabla_\lambda \varphi)^2] \end{aligned} \tag{5}$$