

# NONEXISTENCE RESULT TO DOUBLE PHASE PROBLEMS INVOLVING $\Delta_\lambda$ -LAPLACIAN WITH EXPONENTIAL NONLINEARITY

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**Abstract:** *In this paper, we consider the instability of weak solutions to the problem*

$$-\operatorname{div}_\lambda(w(x)|\nabla_\lambda u|^{p-2}\nabla_\lambda u + \tilde{w}(x)|\nabla_\lambda u|^{q-2}\nabla_\lambda u) = f(x)e^u \text{ in } \mathbb{R}^N,$$

*where  $\nabla_\lambda = (\lambda_1 \partial_{x_1}, \lambda_2 \partial_{x_2}, \dots, \lambda_N \partial_{x_N})$ ,  $\lambda_i, i = 1, \dots, N$  satisfy some general hypotheses,  $q \geq p \geq 2$  and  $w, \tilde{w}, f \in L^1_{\text{loc}}(\mathbb{R}^N)$  are nonnegative functions satisfying some growth conditions at infinity. Our results can be seen as a generalization of that from the Laplace operator to the  $\Delta_\lambda$ -Laplacian. Moreover, our result is also extension of that from the Grushin operator to the  $\Delta_\lambda$ -Laplacian.*

**Key words:** *Double phase problem, exponential nonlinearity, nonexistence result, stable solutions,  $\Delta_\lambda$ -Laplacian.*

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## 1. INTRODUCTION AND MAIN RESULT

In this paper, we define  $\nabla_\lambda = (\lambda_1 \partial_{x_1}, \lambda_2 \partial_{x_2}, \dots, \lambda_N \partial_{x_N})$ ,  $\operatorname{div}_\lambda = \nabla_\lambda \cdot$ ,  $\lambda_i, i = 1, 2, \dots, N$  satisfy some general hypotheses introduced in [1,2]:

(H1) There is a group of dilations  $(\delta_t)_{t>0}$

$$\delta_t: \mathbb{R}^N \rightarrow \mathbb{R}^N, (x_1, \dots, x_N) \mapsto (t^{\varepsilon_1} x_1, \dots, t^{\varepsilon_N} x_N),$$

where  $1 = \varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_N$ , such that

$$\lambda_i(\delta_t(x)) = t^{\varepsilon_i - 1} \lambda_i(x)$$

for all  $x \in \mathbb{R}^N, t > 0$  and  $i = 1, 2, \dots, N$ .

(H2)  $\lambda_1 = 1, \lambda_i(x) = \lambda_i(|x_1|, \dots, |x_{i-1}|)$  for  $i = 2, \dots, N$  and these functions are continuous on  $\mathbb{R}^N$ , strictly positive and of class  $C^1$  on  $\mathbb{R}^N \setminus \Pi$ , where

$$\Pi = \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N : \prod_{i=1}^N x_i = 0 \right\}.$$

Then, the homogeneous dimension of  $\mathbb{R}^N$  associated to the group of dilations  $(\delta_t)_{t>0}$  is

$$Q = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_N \tag{1}$$

and the  $\Delta_\lambda$ -norm of  $x \in \mathbb{R}^N$  is defined by

$$|x|_\lambda = \left( \sum_{j=1}^N \prod_{i \neq j} \lambda_i^2 \varepsilon_j^2 x_j^2 \right)^{\frac{1}{2\sigma}},$$

where  $\sigma = 1 + \sum_{i=1}^N (\varepsilon_i - 1)$ .

The purpose of this paper is to establish nonexistence result of stable weak solutions for the problem

$$-\operatorname{div}_\lambda(w(x)|\nabla_\lambda u|^{p-2}\nabla_\lambda u + \tilde{w}(x)|\nabla_\lambda u|^{q-2}\nabla_\lambda u) = f(x)e^u \text{ in } \mathbb{R}^N \tag{2}$$

Here,  $q \geq p \geq 2$  and  $w, \tilde{w}, f \in L^1_{\text{loc}}(\mathbb{R}^N)$  are nonnegative functions satisfying

$$\begin{cases} w(x) \leq C_1|x|_\lambda^a, \\ \tilde{w}(x) \leq C_2|x|_\lambda^{\tilde{a}}, \\ f(x) \geq C_3|x|_\lambda^b \end{cases} \tag{3}$$

for all  $|x|_\lambda > R_0$ , where  $R_0, C_1, C_2, C_3 > 0$  and  $a, \tilde{a}, b \in \mathbb{R}$ .

Considering the special case  $\lambda_i = 1$ . If  $\tilde{w} = 0, p = 2$ , then (2) becomes the Laplace-type equation. In [3,4], the authors established nonexistence results of stable solutions for the problem

$$-\Delta u = e^u \text{ in } \mathbb{R}^N.$$

In [5], Wang and Ye proved that the equation

$$-\Delta u = |x|^b e^u \text{ in } \mathbb{R}^N,$$

where  $b > -2$ , does not have stable solutions if  $N < 10 + 4b$ . If  $\tilde{w} = 0, p \geq 2$ , then (2) becomes the  $p$ -Laplace-type equation. The results about the instability of solutions for the equation

$$-\Delta_p u = f(x)e^u \text{ in } \mathbb{R}^N$$

can be found in [6]. For the more general  $p$ -Laplace-type equations, we refer to [7–9]. If  $\tilde{w} > 0, q > p \geq 2$ , then (2) becomes the double phase problem. In particular, under the condition

$$N < \frac{\min\{p, q - \tilde{a}\}(q + 3) + 4b}{q - 1},$$

Phuong Le [10] proved that the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u + \tilde{w}(x)|\nabla u|^{q-2}\nabla u) = f(x)e^u \text{ in } \mathbb{R}^N$$

has no stable weak solutions.

Now, we consider the general case of  $\lambda_i$ . In the last decades, there have been many studies on PDEs involving the strongly degenerate elliptic operators  $\Delta_\lambda = \operatorname{div}_\lambda \circ \nabla_\lambda$ , see e.g. [11–18]. They were first introduced by Franchi and Lanconelli in [1] and named  $\Delta_\lambda$ -Laplacians by Kogoj and Lanconelli in [2]. However, to the best of our knowledge, there has no work dealing with Equation (2) in the general case of  $\lambda_i$ . Hence, the purpose of this paper is to establish some results about the instability of weak solutions for Equation (2), where  $\lambda_i$ ,

$i = 1, 2, \dots, N$  satisfy the hypotheses (H1, H2),  $q \geq p \geq 2$  and  $w, \tilde{w}, f \in L^1_{\text{loc}}(\mathbb{R}^N)$  are nonnegative functions satisfying (3).

To state our result, we introduce a functional class of solutions for Equation (2). Let  $\Omega$  be an open domain in  $\mathbb{R}^N$  and put

$$\begin{aligned} H: \Omega \times [0, \infty) &\rightarrow [0, \infty) \\ (x, t) &\mapsto w(x)t^p + \tilde{w}(x)t^q. \end{aligned}$$

Define

$$\rho_H(u) = \int_{\Omega} H(x, |u|) = \int_{\Omega} (w(x)|u|^p + \tilde{w}(x)|u|^q)$$

and

$$L^H(\Omega) = \{u: \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable and } \rho_H(u) < \infty\}.$$

We define the norm in the space  $L^H(\Omega)$  as follows

$$\|u\|_H = \inf \left\{ \tau > 0 \mid \rho_H\left(\frac{u}{\tau}\right) \leq 1 \right\}.$$

The corresponding Sobolev space

$$W^{1,H}(\Omega) = \{u \in L^H(\Omega) \mid |\nabla_{\lambda} u| \in L^H(\Omega)\}$$

is equipped the norm

$$\|u\|_{1,H} = \| |\nabla_{\lambda} u| \|_H + \|u\|_H.$$

We set

$$W^{1,H}_{\text{loc}}(\Omega) = \{u: \Omega \rightarrow \mathbb{R} \mid u\varphi \in W^{1,H}_0(\Omega) \text{ for all } \varphi \in C^1_c(\Omega)\},$$

where  $W^{1,H}_0(\Omega)$  is the closure of  $C^1_c(\Omega)$  in  $W^{1,H}(\Omega)$  with respect to the  $\|\cdot\|_{1,H}$  norm.

In this paper, the solutions of Equation (2) are understood in the sense of weak solutions as follows.

**Definition 1.1.** A function  $u \in W^{1,H}_{\text{loc}}(\mathbb{R}^N)$  is called a weak solution of Equation (2) if  $f(x)e^u \in L^1_{\text{loc}}(\mathbb{R}^N)$  and

$$\int_{\mathbb{R}^N} (w(x)|\nabla_{\lambda} u|^{p-2} \nabla_{\lambda} u + \tilde{w}(x)|\nabla_{\lambda} u|^{q-2} \nabla_{\lambda} u) \cdot \nabla_{\lambda} \varphi = \int_{\mathbb{R}^N} f(x)e^u \varphi \quad (4)$$

for all  $\varphi \in C^1_c(\mathbb{R}^N)$ .

We now define the stable weak solutions of Equation (2).

**Definition 1.2.** A weak solution  $u$  of Equation (2) is called stable if

$$\begin{aligned} &\int_{\mathbb{R}^N} w(x) [|\nabla_{\lambda} u|^{p-2} |\nabla_{\lambda} \varphi|^2 + (p-2) |\nabla_{\lambda} u|^{p-4} (\nabla_{\lambda} u \cdot \nabla_{\lambda} \varphi)^2] \\ &+ \int_{\mathbb{R}^N} \tilde{w}(x) [|\nabla_{\lambda} u|^{q-2} |\nabla_{\lambda} \varphi|^2 \\ &+ (q-2) |\nabla_{\lambda} u|^{q-4} (\nabla_{\lambda} u \cdot \nabla_{\lambda} \varphi)^2] \end{aligned} \quad (5)$$